Lagrangian Approach to Dispersionless KdV Hierarchy

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Abstract. We derive a Lagrangian based approach to study the compatible Hamiltonian structure of the dispersionless KdV and supersymmetric KdV hierarchies and claim that our treatment of the problem serves as a very useful supplement of the so-called r-matrix method. We suggest specific ways to construct results for conserved densities and Hamiltonian operators. The Lagrangian formulation, via Noether's theorem, provides a method to make the relation between symmetries and conserved quantities more precise. We have exploited this fact to study the variational symmetries of the dispersionless KdV equation.

Key words: hierarchy of dispersionless KdV equations; Lagrangian approach; bi-Hamiltonian structure; variational symmetry

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1 Introduction

The equation of Korteweg and de Vries or the so-called KdV equation

$$u_t = \tfrac{1}{4}u_{3x} + \tfrac{3}{2}uu_x$$

in the dispersionless limit [1]

$$\frac{\partial}{\partial t} \to \epsilon\frac{\partial}{\partial t} \qquad \text{and} \qquad \frac{\partial}{\partial x} \to \epsilon\frac{\partial}{\partial x} \qquad \text{with} \quad \epsilon \to 0$$

reduces to

$$u_t = \tfrac{3}{2}uu_x. \tag{1.1}$$

Equation (1.1), often called the Riemann equation, serves as a prototypical nonlinear partial differential equation for the realization of many phenomena exhibited by hyperbolic systems [2]. This might be one of the reasons why, during the last decade, a number of works [3] was envisaged to study the properties of dispersionless KdV and other related equations with special emphasis on their Lax representation and Hamiltonian structure.

The complete integrability of the KdV equation yields the existence of an infinite family of conserved functions or Hamiltonian densities $\mathcal{H}_n$'s that are in involution. All $\mathcal{H}_n$'s that generate flows which commute with the KdV flow give rise to the KdV hierarchy. The equations of the hierarchy can be constructed using [4]

$$u_t = \Lambda^n u_x(x,t), \qquad n = 0, 1, 2, \ldots \tag{1.2}$$

with the recursion operator

$$\Lambda = \tfrac{1}{4}\partial_x^2 + u + \tfrac{1}{2}u_x\partial_x^{-1}.$$

In the dispersionless limit the recursion operator becomes

$$\Lambda = u + \tfrac{1}{2}u_x\partial_x^{-1}. \tag{1.3}$$

According to (1.2), the pseudo-differential operator Λ in (1.3) defines a dispersionless KdV hierarchy. The first few members of the hierarchy are given by

$$n = 0: \qquad u_t = u_x, \tag{1.4a}$$

$$n = 1: \qquad u_t = \tfrac{3}{2}uu_x, \tag{1.4b}$$

$$n = 2: \qquad u_t = \tfrac{15}{8}u^2u_x, \tag{1.4c}$$

$$n = 3: \qquad u_t = \tfrac{35}{16}u^3u_x, \tag{1.4d}$$

$$n = 4: \qquad u_t = \tfrac{315}{128}u^4u_x. \tag{1.4e}$$

Thus the equations in the dispersionless hierarchy can be written in the general form

$$u_t = A_n u^n u_x, \tag{1.5}$$

where the values of A_n should be computed using (1.3) in (1.2). We can also generate A_1, A_2, A_3 etc recursively using

$$A_n = \left(1 + \tfrac{1}{2n}\right)A_{n-1}, \qquad n = 1, 2, 3, \ldots \qquad \text{and} \qquad A_0 = 1.$$

The Hamiltonian structure of the dispersionless KdV hierarchy is often studied by taking recourse to the use of Lax operators expressed in the semi-classical limit [5]. In this work we shall follow a different viewpoint to derive Hamiltonian structure of the equations in (1.5). We shall construct an expression for the Lagrangian density and use the time-honoured method of classical mechanics to rederive and reexamine the corresponding canonical formulation. A single evolution equation is never the Euler–Lagrange equation of a variational problem. One common trick to put a single evolution equation into a variational form is to replace u by a potential function $u = -w_x$. In terms of w, (1.5) will become an Euler–Lagrange equation. We can, however, couple a nonlinear evolution equation with an associated one and derive the action principle. This allows one to write the Lagrangian density in terms of the original field variables rather than the w's, often called the Casimir potential. In Section 2 we adapt both these approaches to obtain the Lagrangian and Hamiltonian densities of the Riemann type equations. In Section 3 we study the bi-Hamiltonian structure [6]. One of the added advantage of the Lagrangian description is that it allows one to establish, via Noether's theorem, the relationship between variational symmetries and associated conservation laws. The concept of variational symmetry results from the application of group methods in the calculus of variations. Here one deals with the symmetry group of an action functional $\mathcal{A}[u] = \int_{\Omega_0}\mathcal{L}\left(x, u^{(n)}\right)dx$ with $\mathcal{L}$, the so-called Lagrangian density of the field $u(x)$. The groups considered will be local groups of transformations acting on an open subset $\mathcal{M} \subset \Omega_0 \times U \subset X \times U$. The symbols X and U denote the space of independent and dependent variables respectively. We devote Section 4 to study this classical problem. Finally, in Section 5 we make some concluding remarks.

2 Lagrangian and Hamiltonian densities

For $u = -w_x$ (1.5) becomes

$$w_{xt} = A_n(-1)^n w_x^n w_{2x}. \tag{2.1}$$

The Fréchet derivative of the right side of (2.1) is self-adjoint. Thus we can use the homotopy formula [7] to obtain the Lagrangian density in the form

$$\mathcal{L}_n = \tfrac{1}{2} w_t w_x + \frac{A_n(-1)^{n+1}}{(n+1)(n+2)} w_x^{n+2}. \tag{2.2}$$

In writing (2.2) we have subtracted a gauge term which is harmless at the classical level. The subscript n of $\mathcal{L}$ merely indicates that it is the Lagrangian density for the nth member of the dispersionless KdV hierarchy. The corresponding canonical Hamiltonian densities obtained by the use of Legendre map are given by

$$\mathcal{H}_n = \frac{A_n}{(n+1)(n+2)} u^{n+2}. \tag{2.3}$$

Equation (1.5) can be written in the form

$$u_t + \frac{\partial \rho[u]}{\partial x} = 0 \tag{2.4}$$

with

$$\rho[u] = -\frac{A_n}{(n+1)} u^{n+1}. \tag{2.5}$$

There exists a prolongation of (1.5) or (2.4) into another equation

$$v_t + \frac{\delta(\rho[u] v_x)}{\delta u} = 0, \qquad v = v(x,t) \tag{2.6}$$

with the variational derivative

$$\frac{\delta}{\delta u} = \sum_{k=0}^{m} (-1)^k \frac{\partial^k}{\partial x^k} \frac{\partial}{\partial u_{kx}}, \qquad u_{kx} = \frac{\partial^k u}{\partial x^k}$$

such that the coupled system of equations follows from the action principle [8]

$$\delta \int \mathcal{L}^c \, dx dt = 0.$$

The Lagrangian density for the coupled equations in (2.4) and (2.6) is given by

$$\mathcal{L}^c = \tfrac{1}{2}(v u_t - u v_t) - \rho[u] v_x.$$

For $\rho[u]$ in (2.5), (2.6) reads

$$v_t = A_n u^n v_x. \tag{2.7}$$

For the system represented by (1.5) and (2.7) we have

$$\mathcal{L}_n^c = \tfrac{1}{2}(v u_t - u v_t) + \frac{A_n}{(n+1)} u^{n+1} v_x. \tag{2.8}$$

The result in (2.7) could also be obtained using the method of Kaup and Malomed [9]. Referring back to the supersymmetric KdV equation [10] we identify v as a fermionic variable associated with the bosonic equation in (1.5). It is of interest to note that the supersymmetric system is complete in the sense of variational principle while neither of the partners is. The Hamiltonian density obtained from the Lagrangian in (2.8) is given by

$$\mathcal{H}_n^c = -\frac{A_n}{(n+1)} u^{n+1} v_x. \tag{2.9}$$

It remains an interesting curiosity to demonstrate that the results in (2.3) and (2.9) represent the conserved densities of the dispersionless KdV and supersymmetric KdV flows. We demonstrate this by examinning the appropriate bi-Hamiltonian structures of (1.5) and the pair (1.5) and (2.7).

3 Bi-Hamiltonian structure

Zakharov and Faddeev [11] developed the Hamiltonian approach to integrability of nonlinear evolution equations in one spatial and one temporal (1+1) dimensions and Gardner [12], in particular, interpreted the KdV equation as a completely integrable Hamiltonian system with ∂_x as the relevant Hamiltonian operator. A significant development in the Hamiltonian theory is due to Magri [6] who realized that integrable Hamiltonian systems have an additional structure. They are bi-Hamiltonian, i.e., they are Hamiltonian with respect to two different compatible Hamiltonian operators. A similar consideration will also hold good for the dispersionless KdV equations and we have

$$u_t = \partial_x \left(\frac{\delta H_n}{\delta u} \right) = \tfrac{1}{2} \left(u\partial_x + \partial_x u \right) \left(\frac{\delta H_{n-1}}{\delta u} \right), \qquad n = 1, 2, 3 \ldots . \tag{3.1}$$

Here

$$H = \int \mathcal{H} dx. \tag{3.2}$$

It is easy to verify that for $n = 1$, (2.3), (3.1) and (3.2) give (1.4b). The other equations of the hierarchy can be obtained for $n = 2, 3, 4, \ldots$. The operators $\mathcal{D}_1 = \partial_x$ and $\mathcal{D}_2 = \frac{1}{2}\left(u\partial_x + \partial_x u \right)$ in (3.1) are skew-adjoint and satisfy the Jacobi identity. The dispersionless KdV equation, in particular, can be written in the Hamiltonian form as

$$u_t = \{u(x), H_1\}_1 \qquad \text{and} \qquad u_t = \{u(x), H_0\}_2$$

endowed with the Poisson structures

$$\{u(x), u(y)\}_1 = \mathcal{D}_1 \delta(x-y) \qquad \text{and} \qquad \{u(x), u(y)\}_2 = \mathcal{D}_2 \delta(x-y).$$

Thus $\mathcal{D}_1$ and $\mathcal{D}_2$ constitute two compatible Hamiltonian operators such that the equations obtained from (1.5) are integrable in Liouville's sense [6]. Thus $\mathcal{H}_n$'s in (2.3) via (3.2) give the conserved densities of (1.5). In other words, $\mathcal{H}_n$'s generate flows which commute with the dispersionless KdV flow and give rise to an appropriate hierarchy. It will be quite interesting to examine if a similar analysis could also be carried out for the supersymmetric dispersionless KdV equations.

The pair of supersymmetric equations $u_t = u^n u_x$ and $v_t = u^n v_x$ can be written as

$$\eta_t = \mathbf{J}_1 \left(\frac{\delta H_n^s}{\delta \eta} \right) = \mathbf{J}_2 \left(\frac{\delta H_{n-1}^s}{\delta \eta} \right), \tag{3.3}$$

where $\eta = \begin{pmatrix} u \\ v \end{pmatrix}$, $H_n^s = \frac{H_n^c}{A_n}$ and $H_n^c = \int \mathcal{H}_n^c dx$. In (3.3) $\mathbf{J}_1$ and $\mathbf{J}_2$ stand for the matrices

$$\mathbf{J}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{J}_2 = \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix}. \tag{3.4}$$

Since H_n^c for different values of n represent the conserved Hamiltonian densities obtained by the use of action principle, the supersymmetric dispersionless KdV equations will be bi-Hamiltonian provided $\mathbf{J}_1$ and $\mathbf{J}_2$ constitute a pair of compatible Hamiltonian operators. Clearly, $\mathbf{J}_1$ and $\mathbf{J}_2$ are skew-adjoint. Thus $\mathbf{J}_1$ and $\mathbf{J}_2$ will be Hamiltonian operators provided we can show that [5]

$$\mathrm{pr}\, \mathrm{v}_{\mathbf{J}_i \theta}(\Theta_{\mathbf{J}_i}) = 0, \qquad i = 1, 2. \tag{3.5}$$

Here pr stands for the prolongation of the evolutionary vector field v of the characteristic $\mathbf{J}_i\theta$. The quantity $\mathrm{pr}\,\mathrm{v}_{\mathbf{J}_i}\theta$ is calculated by using

$$\mathrm{pr}\,\mathrm{v}_{\mathbf{J}_i\theta} = \sum_{\mu,j} D_j \left(\sum_{\nu} (\mathbf{J}_i)_{\mu\nu}\theta^{\nu} \right) \frac{\partial}{\partial \eta^{\mu}_j}, \qquad D_j = \frac{\partial}{\partial x^j}, \qquad \mu,\nu = 1,2. \tag{3.6}$$

In our case the column matrix $\theta = \begin{pmatrix} \phi \\ \psi \end{pmatrix}$ represents the basis univectors associated with the variables $\eta = \begin{pmatrix} u \\ v \end{pmatrix}$. Understandably, θ^{ν} and η^{μ} denote the components of θ and η and $(\mathbf{J}_i)_{\mu\nu}$ carries a similar meaning. The functional bivectors corresponding to the operators $\mathbf{J}_i$ is given by

$$\Theta_{\mathbf{J}_i} = \tfrac{1}{2} \int \theta^T \wedge \mathbf{J}_i \theta dx \tag{3.7}$$

with θ^T, the transpose of θ. From (3.4), (3.6) and (3.7) we found that both $\mathbf{J}_1$ and $\mathbf{J}_2$ satisfy (3.5) such that each of them constitutes a Hamiltonian operator. Further, one can check that $\mathbf{J}_1$ and $\mathbf{J}_2$ satisfy the compatibility condition

$$\mathrm{pr}\,\mathrm{v}_{\mathbf{J}_1\theta}(\Theta_{\mathbf{J}_2}) + \mathrm{pr}\,\mathrm{v}_{\mathbf{J}_2\theta}(\Theta_{\mathbf{J}_1}) = 0.$$

This shows that (3.3) gives the bi-Hamiltonian form of supersymmetric dispersionless KdV equations. The recursion operator defined by

$$\Lambda = \mathbf{J}_2\mathbf{J}_1^{-1} = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}$$

reproduces the hierarchy of supersymmetric dispersionless KdV equation according to

$$\eta_t = \Lambda^n \eta_x.$$

for $n = 0, 1, 2, \ldots$. This verifies that $\frac{\mathcal{H}^c_n}{A_n}$'s as conserved densities generate flows which commute with the supersymmetric dispersionless KdV flow.

4 Variational symmetries

The Lagrangian and Hamiltonian formulations of dynamical systems give a way to make the relation between symmetries and conserved quantities more precise and thereby provide a method to derive expressions for the conserved quantities from the symmetry transformations. In its general form this is referred to as Noether's theorem. More precisely, this theorem asserts that if a given system of differential equations follows from the variational principle, then a continuous symmetry transformation (point, contact or higher order) that leaves the action functional invariant to within a divergence yields a conservation law. The proof of this theorem requires some knowledge of differential forms, Lie derivatives and pull-back [5]. We shall, however, carry out the symmetry analysis for the dispersionless KdV equation using a relatively simpler mathematical framework as compared to that of the algebro-geometric theories. In fact, we shall make use of some point transformations that depend on time and spatial coordinates. The approach to be followed by us has an old root in the classical-mechanics literature. For example, as early as 1951, Hill [13] provided a simplified account of Noether's theorem by considering infinitesimal transformations of the dependent and independent variables characterizing the classical field. We shall first present our general scheme for symmetry analysis and then study the variational or Noether's symmetries of the dispersionless KdV equation.

Consider the infinitesimal transformations

$$x^{i'} = x^i + \delta x^i, \qquad \delta x^i = \epsilon \xi^i(x, f) \tag{4.1a}$$

and

$$f' = f + \delta f, \qquad \delta f = \epsilon \eta(x, f) \tag{4.1b}$$

for a field variable $f = f(x, t)$ with ϵ, an arbitrary small quantity. Here $x = \{x^0, x^1\}$, $x^0 = t$ and $x^1 = x$. Understandably, our treatment for the symmetry analysis will be applicable to $(1+1)$ dimensional cases. However, the result to be presented here can easily be generalized to deal with $(3+1)$ dimensional problems. For an arbitrary analytic function $g = g(x^i, f)$, it is straightforward to show that

$$\delta g = \epsilon X g$$

with

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial f}, \tag{4.2}$$

the generator of the infinitesimal transformations in (4.1). A similar consideration when applied to $h = h(x^i, f, f_i)$ with $f_i = \frac{\partial f}{\partial x^i}$ gives

$$\delta h = \epsilon X' h \tag{4.3}$$

with

$$X' = X + \left(\eta_i - \xi_i^j f_j\right) \frac{\partial}{\partial f_i}. \tag{4.4}$$

Understandably, X' stands for the first prolongation of X. To arrive at the statement for the Noether's theorem we consider among the general set of transformations in (4.1) only those that leave the field-theoretic action invariant. We thus write

$$\mathcal{L}(x^i, f, f_i) d(x) = \mathcal{L}'(x^{i'}, f', f_i{}') d(x'), \tag{4.5}$$

where $d(x) = dxdt$. In order to satisfy the condition in (4.5) we allow the Lagrangian density to change its functional form $\mathcal{L}$ to $\mathcal{L}'$. If the equations of motion, expressed in terms of the new variables, are to be of precisely the same functional form as in the old variables, the two density functions must be related by a divergence transformation. We thus express the relation between $\mathcal{L}'$ and $\mathcal{L}$ by introducing a gauge function $B^i(x, f)$ such that

$$\mathcal{L}'(x^{i'}, f', f_i{}') d(x') = \mathcal{L}(x^{i'}, f', f_i{}') d(x') - \epsilon \frac{dB^i}{dx^{i'}} d(x') + o(\epsilon^2). \tag{4.6}$$

The general form of (4.6) for the definition of symmetry transformations will allow the scale and divergence transformations to be considered as symmetry transformations. Understandably, the scale transformations give rise to Noether's symmetries while the scale transformations in conjunction with the divergence term lead to Noether's divergence symmetries. Traditionally, the concept of divergence symmetries and concommitant conservation laws are introduced by replacing Noether's infinitesimal criterion for invariance by a divergence condition [14]. However, one can directly work with the conserved densities that follow from (4.6) because nature of the vector fields will determine the contributions of the gauge term. For some of the vector fields the contributions of B^i to conserved quantities will be equal to zero. These vector fields are

Noether's symmetries else we have Noether's divergence symmetries. In view of (4.5), (4.6) can be written in the form

$$\mathcal{L}(x^{i\prime}, f', f_i{}')d(x') = \mathcal{L}(x^i, f, f_i)d(x) + \epsilon \frac{dB^i}{dx^i} d(x). \tag{4.7}$$

Again using $\mathcal{L}$ for h in (4.3), we have

$$\mathcal{L}(x^{i\prime}, f', f_i{}')d(x') = \mathcal{L}(x^i, f, f_i)\left[d(x) + \epsilon d\xi^i(x, f_i)\right] + \epsilon X' \mathcal{L}(x^i, f, f_i)d(x). \tag{4.8}$$

From (4.7) and (4.8), we write

$$\frac{dB^i}{dx^i} = \frac{d\xi^i}{dx^i}\mathcal{L} + X'\mathcal{L}. \tag{4.9}$$

Using the value of X' from (4.4) in (4.9), $\frac{dB^i}{dx^i}$ is obtained in the final form

$$\frac{dB^i}{dx^i} = \frac{d\xi^i}{dx^i}\mathcal{L} + \xi^i \frac{\partial \mathcal{L}}{\partial x^i} + \eta \frac{\partial \mathcal{L}}{\partial f} + \left(\eta_i - \xi_i^j f_j\right)\frac{\partial \mathcal{L}}{\partial f_i}. \tag{4.10}$$

Thus we find that the action is invariant under those transformations whose constituents ξ and η satisfy (4.10). The terms in (4.10) can be rearranged to write

$$\frac{d}{dx^i}\left\{B^i - \xi^i\mathcal{L} + \left(\xi^j f_j - \eta\right)\frac{\partial \mathcal{L}}{\partial f_i}\right\} + \left(\xi^j f_j - \eta\right)\left[\frac{\partial \mathcal{L}}{\partial f} - \frac{d}{dx^i}\left(\frac{\partial \mathcal{L}}{\partial f_i}\right)\right] = 0. \tag{4.11}$$

The expression inside the squared bracket stands for the Euler–Lagrange equation for the classical field under consideration. In view of this, (4.11) leads to the conservation law

$$\frac{d\mathcal{I}^i}{dx^i} = 0 \tag{4.12}$$

with the conserved density given by

$$\mathcal{I}^i = B^i - \xi^i\mathcal{L} + \left(\xi^j f_j - \eta\right)\frac{\partial \mathcal{L}}{\partial f_i}. \tag{4.13}$$

In the case of two independent variables $(x^0, x^1) \equiv (t, x)$, (4.12) can be written in the explicit form

$$\frac{d\mathcal{I}^0}{dt} + \frac{d\mathcal{I}^1}{dx} = 0. \tag{4.14}$$

From (2.2) the Lagrangian density for the dispersionless KdV equation is obtained as

$$\mathcal{L} = \tfrac{1}{2} w_t w_x + \tfrac{1}{4} w_x^3. \tag{4.15}$$

Identifying f with w we can combine (4.13), (4.14) and (4.15) to get

$$\begin{aligned} &B_t^0 + w_t B_w^0 - \tfrac{1}{4}\xi_t^0 w_x^3 - \tfrac{1}{4}\xi_w^0 w_t w_x^3 + \tfrac{1}{2}\xi_t^1 w_x^2 + \tfrac{1}{2}\xi_w^1 w_t w_x^2 - \tfrac{1}{2}\eta_t w_x - \eta_w w_t w_x \\ &\quad + B_x^1 + w_x B_w^1 + \tfrac{1}{2}\xi_x^1 w_x^3 + \tfrac{1}{2}\xi_w^1 w_x^4 + \tfrac{1}{2}\xi_x^0 w_t^2 + \tfrac{1}{2} w_x w_t^2 \xi_w^0 + \tfrac{3}{4}\xi_x^0 w_x^2 w_t - \tfrac{3}{4}\eta_x w_x^2 \\ &\quad - \tfrac{1}{2}\eta_x w_t - \tfrac{3}{4}\eta_w w_x^3 + \tfrac{3}{4}\xi_w^0 w_x^3 w_t = 0. \end{aligned} \tag{4.16}$$

In writing (4.16) we have made use of (2.1) with $n = 1$. Equation (4.16) can be globally satisfied iff the coefficients of the following terms vanish separately

$$w_x^0 \quad \text{or} \quad w_t^0 : \qquad B_t^0 + B_x^1 = 0, \tag{4.17a}$$
$$w_t : \qquad B_w^0 - \tfrac{1}{2}\eta_x = 0, \tag{4.17b}$$
$$w_t^2 : \qquad \tfrac{1}{2}\xi_x^0 = 0, \tag{4.17c}$$
$$w_x : \qquad B_w^1 - \tfrac{1}{2}\eta_t = 0, \tag{4.17d}$$
$$w_x^2 : \qquad \tfrac{1}{2}\xi_t^1 - \tfrac{3}{4}\eta_x = 0, \tag{4.17e}$$
$$w_x^3 : \qquad -\tfrac{1}{4}\xi_t^0 - \tfrac{3}{4}\eta_w + \tfrac{1}{2}\xi_x^1 = 0, \tag{4.17f}$$
$$w_x^4 : \qquad \tfrac{1}{2}\xi_w^1 = 0, \tag{4.17g}$$
$$w_t w_x : \qquad -\eta_w = 0, \tag{4.17h}$$
$$w_t w_x^2 : \qquad \tfrac{1}{2}\xi_w^1 + \tfrac{3}{4}\xi_x^0 = 0, \tag{4.17i}$$
$$w_t w_x^3 : \qquad \tfrac{1}{2}\xi_w^0 = 0, \tag{4.17j}$$
$$w_t^2 w_x : \qquad \tfrac{1}{2}\xi_w^0 = 0. \tag{4.17k}$$

Equations in (4.17) will lead to finite number of symmetries. This number appears to be disappointingly small since we have a dispersionless KdV hierarchy given in (1.5). Further, symmetry properties reflecting the existence of infinitely many conservation laws will require an appropriate development for the theory of generalized symmetries. In this work, however, we shall be concerned with variational symmetries only.

From (4.17c), (4.17j) and (4.17k) we see that ξ^0 is only a function of t. We, therefore, write

$$\xi^0(x, t, w) = \beta(t). \tag{4.18}$$

Also from (4.17g), (4.17i) and (4.18) we see that ξ^1 is not a function of w. In view of (4.17h) and (4.18), (4.17f) gives

$$\xi_x^1 - \tfrac{1}{2}\beta_t = 0$$

which can be solved to get

$$\xi^1 = \tfrac{1}{2}\beta_t x + \alpha(t), \tag{4.19}$$

where $\alpha(t)$ is a constant of integration. Using (4.19) in (4.17e) we have

$$\eta_x = \tfrac{1}{3}\beta_{tt} x + \tfrac{2}{3}\alpha_t. \tag{4.20}$$

The solution of (4.20) is given by

$$\eta = \tfrac{1}{6}\beta_{tt} x^2 + \tfrac{2}{3}\alpha_t x + \gamma(t) \tag{4.21}$$

with $\gamma(t)$, a constant of integration. In view of (4.21), (4.17b) and (4.17d) yield

$$B^0 = \tfrac{1}{6}\beta_{tt} x w + \tfrac{1}{3}\alpha_t w \tag{4.22}$$

and

$$B^1 = \tfrac{1}{12}\beta_{ttt} x^2 w + \tfrac{1}{3}\alpha_{tt} x w. \tag{4.23}$$

Equations (4.22) and (4.23) can be combined with (4.17a) to get finally

$$\beta_{ttt} = 0 \qquad \text{and} \qquad \alpha_{tt} = 0. \tag{4.24}$$

From (4.24) we write

$$\beta = \tfrac{1}{2}a_1 t^2 + a_2 t + a_3 \tag{4.25}$$

and

$$\alpha = b_1 t + b_2, \tag{4.26}$$

where a's and b's are arbitrary constants. Substituting the values of β and α in (4.18), (4.19), (4.21) we obtain the infinitesimal transformation, ξ^0, ξ^1 and η, as

$$\xi^0 = \tfrac{1}{2}a_1 t^2 + a_2 t + a_3, \tag{4.27a}$$

$$\xi^1 = \tfrac{1}{2}(a_1 t + a_2)x + b_1 t + b_2, \tag{4.27b}$$

$$\eta = \tfrac{1}{6}a_1 x^2 + \tfrac{2}{3}b_1 x + b_3. \tag{4.27c}$$

In writing (4.27c) we have treated $\gamma(t)$ as a constant and replaced it by b_3. Implication of this choice will be made clear while considering the symmetry algebra. In terms of (4.27), (4.2) becomes

$$X = a_1 V_1 + a_2 V_2 + a_3 V_3 + b_1 V_4 + b_2 V_5 + b_3 V_6,$$

where

$$V_1 = \tfrac{1}{2}t^2\frac{\partial}{\partial t} + \tfrac{1}{2}xt\frac{\partial}{\partial x} + \tfrac{1}{6}x^2\frac{\partial}{\partial w}, \qquad V_2 = t\frac{\partial}{\partial t} + \tfrac{1}{2}x\frac{\partial}{\partial x},$$
$$V_3 = \frac{\partial}{\partial t}, \qquad V_4 = t\frac{\partial}{\partial x} + \tfrac{2}{3}x\frac{\partial}{\partial w}, \qquad V_5 = \frac{\partial}{\partial x}, \qquad V_6 = \frac{\partial}{\partial w}. \tag{4.28}$$

It is easy to check that the vector fields $V_1, \ldots, V_6$ satisfy the closure property. The commutation relations between these vector fields are given in Table 1.

Table 1. Commutation relations for the generators in (4.28). Each element V_{ij} in the Table is represented by $V_{ij} = [V_i, V_j]$.

	V_1	V_2	V_3	V_4	V_5	V_6
V_1	0	$-V_1$	$-V_2$	0	$-\frac{1}{2}V_4$	0
V_2	V_1	0	$-V_3$	$\frac{1}{2}V_4$	$-\frac{1}{2}V_5$	0
V_3	V_2	V_3	0	V_5	0	0
V_4	0	$-\frac{1}{2}V_4$	$-V_5$	0	$-\frac{2}{3}V_6$	0
V_5	$\frac{1}{2}V_4$	$\frac{1}{2}V_5$	0	$\frac{2}{3}V_6$	0	0
V_6	0	0	0	0	0	0

The symmetries in (4.28) are expressed in terms of the velocity field and depend explicitly on x and t. Looking from this point of view the symmetry vectors obtained by us bear some similarity with the so called 'addition symmetries' suggested independently by Chen, Lee and Lin [15] and by Orlov and Shulman [16]. It is easy to see that V_2 to V_6 correspond to scaling, time translation, Galilean boost, space translation and translation in velocity space respectively. The vector field V_1 does not admit such a simple physical realization. However, we can write V_1 as $V_1 = \frac{1}{2}tV_2 + \frac{1}{4}xV_4$.

Making use of (4.15), (4.22), (4.23), (4.25) and (4.26) we can write the expressions for the conserved quantities in (4.13) as

$$\mathcal{I}^0 = \tfrac{1}{6}a_1 xw + \tfrac{1}{3}b_1 w - \tfrac{1}{4}\xi^0 w_x^3 + \tfrac{1}{2}\xi^1 w_x^2 - \tfrac{1}{2}\eta w_x, \tag{4.29a}$$

$$\mathcal{I}^1 = \tfrac{1}{2}\xi^0 w_t^2 + \tfrac{3}{4}\xi^0 w_t w_x^2 + \tfrac{1}{2}\xi^1 w_x^3 - \tfrac{1}{2}\eta w_t - \tfrac{3}{4}\eta w_x^2. \tag{4.29b}$$

The expressions for $\mathcal{I}^0$ and $\mathcal{I}^1$ are characterized by ξ^i and η, the values of which change as we go from one vector field to the other. The first two terms in $\mathcal{I}^0$ stand for the contribution of B^0 and there is no contribution of the gauge term in $\mathcal{I}^1$ since from (4.23) and (4.24) $B^1 = 0$. For a particular vector field a_1 and b_1 may either be zero or non zero. One can verify that except for vector fields V_1 and V_4, $a_1 = b_1 = 0$ such that V_2, V_3, V_5 and V_6 are simple Noether's symmetries while V_1 and V_4 are Noether's divergence symmetries. Coming down to details we have found the following conserved quantities from (4.29a) and (4.29b)

$$\mathcal{I}^0_{V_1} = \tfrac{1}{6}xw - \tfrac{1}{8}t^2w_x^3 + \tfrac{1}{4}xtw_x^2 - \tfrac{1}{12}x^2w_x, \tag{4.30a}$$

$$\mathcal{I}^1_{V_1} = \tfrac{1}{4}xtw_x^3 + \tfrac{3}{8}t^2w_tw_x^2 + \tfrac{1}{4}t^2w_t^2 - \tfrac{1}{12}x^2w_t - \tfrac{1}{8}x^2w_x^2, \tag{4.30b}$$

$$\mathcal{I}^0_{V_2} = -\tfrac{1}{4}tw_x^3 + \tfrac{1}{4}xw_x^2, \tag{4.30c}$$

$$\mathcal{I}^1_{V_2} = \tfrac{1}{4}xw_x^3 + \tfrac{3}{4}tw_tw_x^2 + \tfrac{1}{2}tw_t^2, \tag{4.30d}$$

$$\mathcal{I}^0_{V_3} = -\tfrac{1}{4}w_x^3, \tag{4.30e}$$

$$\mathcal{I}^1_{V_3} = \tfrac{3}{4}w_tw_x^2 + \tfrac{1}{2}w_t^2, \tag{4.30f}$$

$$\mathcal{I}^0_{V_4} = \tfrac{1}{3}w + \tfrac{1}{2}tw_x^2 - \tfrac{1}{3}xw_x, \tag{4.30g}$$

$$\mathcal{I}^1_{V_4} = \tfrac{1}{2}tw_x^3 - \tfrac{1}{3}xw_t - \tfrac{1}{2}xw_x^2, \tag{4.30h}$$

$$\mathcal{I}^0_{V_5} = \tfrac{1}{2}w_x^2, \tag{4.30i}$$

$$\mathcal{I}^1_{V_5} = \tfrac{1}{2}w_x^3, \tag{4.30j}$$

$$\mathcal{I}^0_{V_6} = -\tfrac{1}{2}w_x, \tag{4.30k}$$

$$\mathcal{I}^1_{V_6} = -\tfrac{1}{2}w_t - \tfrac{3}{4}w_x^2. \tag{4.30l}$$

It is easy to check that the results in (4.30) is consistent with (4.14). The pair of conserved quantities corresponding to time translation, space translation and velocity space translation, namely, {(4.30e),(4.30f)}, {(4.30i),(4.30j)} and {(4.30k),(4.30l)} do not involve x and t explicitly. Each of the pair in conjunction with (4.14) give the dispersionless KdV equation in a rather straightforward manner. As expected (4.30e) stands for the Hamiltonian density or energy of (1.4b).

5 Conclusion

Compatible Hamiltonian structures of the dispersionless KdV hierarchy are traditionally obtained with special attention to their Lax representation in the semiclassical limit. The derivation involves judicious use of the so-called r-matrix method [17]. We have shown that the combined Lax representation–r-matrix method can be supplemented by a Lagrangian approach to the problem. We found that the Hamiltonian densities corresponding to our Lagrangian representations stand for the conserved densities for the dispersionless KdV flow. We could easily construct the Hamiltonian operators from the recursion operator which generates the hierarchy. We have derived the bi-Hamiltonian structures for both dispersionless KdV and supersymmetric KdV hierarchies. As an added realism of the Lagrangian approach we studied the variational symmetries of equation (1.4b). We believe that it will be quite interesting to carry out similar analysis for the supersymmetric KdV pair in (1.4b) and for $n = 1$ limit of (2.7).

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